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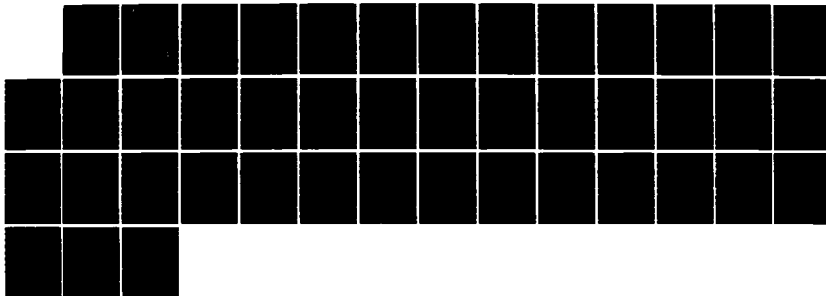
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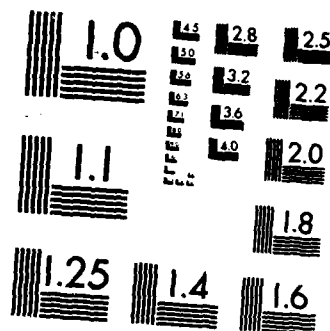
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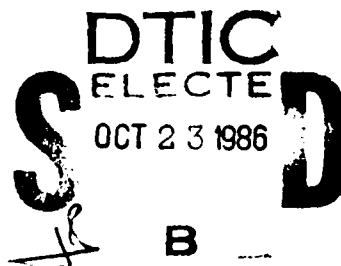
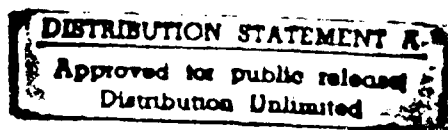
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THE p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD.  
THE STATE OF THE ART

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# THE p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD.

## THE STATE OF THE ART

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### 1. INTRODUCTION

There are three versions of the finite element method. The classical h-version, which achieves the accuracy by refining the mesh while using low degrees  $p$  of the elements, usually  $p = 1, 2$ . The p-version keeps the mesh fixed and the accuracy is achieved by increasing the degree  $p$ . The h-p version combines both approaches.

The  $p$  and  $h-p$  versions are new developments. The  $p$ -version was implemented at Washington University in St. Louis in an experimental code called COMET-X in the middle of 1970. The essential part of the code were hierarchic elements. This type of elements was first considered by Zienkiewicz, Irons, Scott and Campbell [1970] in conjunction with joining finite elements of different polynomial degrees. Hierarchic  $C^0$  elements were then described by various authors, e.g. Peano [1975], Katz, Peano and Rossow [1978], Szabó and Peano [1983], Zienkiewicz, Gago and Kelly [1983]. The cohesive description of the  $p$ -version has been given in Szabó [1979].

The first theoretical analysis of the  $p$ -version was given in Babuška, Szabó and Katz [1981]; and in Babuška and Szabó [1982]. The performance of the  $h-p$  version was first theoretically studied in Babuška and Dorr [1981]. For theoretical analysis of the  $p$ -version in 3-dimensions, we refer to Dorr [1984] and Dorr [1986]. Additional recent results are mentioned below.

For the implementational aspects of the  $p$ -version, we refer to Szabó [1985], Szabó [1986] and Szabó [1986a].

The  $p$  and  $h-p$  versions for two dimensional problems were implemented in the commercial system PROBE by Noetic Tech., St. Louis with first release in 1985, and the second one in 1986 (computations in the present paper are made by PROBE). The three dimensional finite element code FIESTA having some  $p$ -version capabilities was developed at ISMES (Istituto Sperimentali Modelli e Strutture) in Bergamo, Italy, and since early 1980 this program has been available in USA. A new implementation for three dimensional applications on Cray computers was begun by the Aeronautical Research Institute of Sweden (Glyttekniska Försöksanstalten FFA). The  $p$  and  $h-p$  versions are used in the industry today.

Although the p and h-p versions of the finite element method are relatively new developments, many basic results are available. The aim of the ~~present~~ paper is to give a survey of the basic available results and directions for further development. The paper tries to survey basic theoretical implementational and computational aspects of the method as of today.

## 2. FINITE ELEMENT METHOD AND THE APPROXIMATION PROBLEM

Let  $B(u,v)$  be a bilinear form defined on  $H_1 \times H_2$ , where  $H_1$  and  $H_2$  be reflexive Banach spaces equipped with the norm  $|\cdot|_1$  and  $|\cdot|_2$ , respectively. Let further  $F \in H_2^*$ , i.e.  $F$  be a linear functional on  $H_2$ .

By the problem  $\{B,F\}$  we denote the problem to find  $u_0 \in H_1$  so that

$$B(u_0, v) = F(v) \quad (2.1)$$

holds for all  $v \in H_2$ .

If the bilinear form  $B(u,v)$  is continuous and satisfies the so called inf-sup condition (see Babuška, Aziz [1972], Ch. 5), then the problem  $\{B,F\}$  has unique solution.

Let now  $S_1 \in H_1$ ,  $S_2 \in H_2$ . Then the finite element problem  $\{B,F,S_1,S_2\}$  is to find the finite element solution  $u_{S_1} \in S_1$  such that

$$B(u_{S_1}, v) = F(v) = B(u_0, v) \quad (2.2)$$

holds for all  $v \in S_2$ .

If the bilinear form  $B(u,v)$  satisfies the inf sup condition on  $S_1 \times S_2$ , then  $u_{S_1}$  exists, is unique and

$$|u_{S_1} - u_0|_1 \leq C(S_1, S_2) Z(u_0, H_1, S_1) \quad (2.3)$$

where

$$Z(u_0, H_1, S_1) = \inf_{w \in S_1} |u_0 - w|_1. \quad (2.4)$$

For detail, see Babuška, Aziz [1972, Ch. 6] and Arnold, Babuška, Osborn [1985]. We will assume that

$$C(S_1, S_2) \leq D \quad (2.5)$$

where  $D$  is independent of  $S_1, S_2$  and hence the norm of the error  $e = u_{S_1} - u_0$  of the finite element solution is completely given by  $Z(u_0, H_1, S_1)$

Remark 2.1. We do not need necessarily that (2.5) holds. Nevertheless, assumption (2.5) simplifies our conclusions.

Remark 2.2. The condition (2.5) is satisfied, for example, if  $H_1 = H_2$ ,  $B(u,v) = B(v,u)$ ,  $S_1 = S_2$  and



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$$B(u,u) \geq \gamma \|u\|_1^2,$$

$$\gamma > 0.$$

This condition is satisfied for selfadjoint positive definite problems as in the theory of elasticity, etc.  $\Delta$

The exact solution  $u_0$  is, of course, not known. Nevertheless, we will assume that it is a-priori known that  $u_0 \in K \subset H_1$ , where  $K$  is a certain set called the solution set which is compact in  $H_1$ . We define

$$Z(K, H_1, S_1) = \sup_{u \in K} Z(u, H_1, S_1) \quad (2.6)$$

which characterizes the error under the assumption that we know only that the solution  $u_0 \in K$ .

Remark 2.3. A typical example is that  $H_1 = H^1(\Omega)$  and

$$K = \{u \mid \|u\|_{H^k(\Omega)} \leq 1, \quad k > 1\}.$$

This choice leads then to the classical error estimate of the error of the finite element solution (h-version):

$$\|e\|_{H^1} \leq Ch^k \|u_0\|_{H^k}.$$

See, e.g., Babuška, Aziz [1972, Ch. 4].  $\Delta$

There are many results available concerning the characterization of  $Z(K, H_1, S_1)$ , the best selection of  $S_1$  of dimension  $n$ , etc. For an excellent abstract treatment and survey of available results, we refer to Pinkus [1985].

The space  $S_1$  in (2.6) is fully characterized by the finite element method, its  $h$ ,  $p$  or  $h$ - $p$  versions. The set  $K$  is characterized by the class of problems to be solved. Hence, the performance of the finite element method relatively to the solution set  $K$  is described by  $Z(K, H, S)$  which will be in the center of our interest. Of course, others aspects are also essential for the assessment of the performance of the finite element method. They will also be described in this survey.

### 3. THE MODEL PROBLEM

The performance of the method depends strongly on the class of problems for which it is used. As said in Section 2, the performance is directly related to the solution set  $K$  under consideration. We will concentrate here especially on the class of problems which are characterized by the piecewise analytic input data.

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain and its boundary  $\partial\Omega$  be a piecewise analytic curve  $\Gamma = \bigcup_{i=1}^n \Gamma_i$  where  $\Gamma_i$  are (closed) arcs with the end points  $A_i, A_{i+1}$ ,  $i = 1, \dots, n$  ( $A_{n+1} = A_1$ ). An example and the notation is shown in Fig. 3.1.

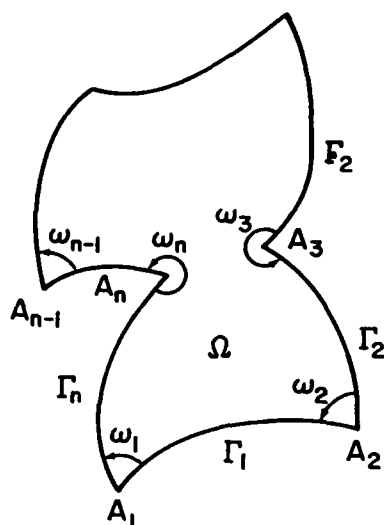


Fig. 3.1. The scheme of a domain with piecewise analytic boundary.

By  $A_i$ ,  $i = 1, 2, \dots, n$  we denote the vertices of  $\Omega$  and by  $\omega_i$ ,  $i = 1, 2, \dots, n$  the internal angles. We will not exclude the case when the internal angle  $\omega = 2\pi$ . This case is very important in practice (cracks) when two arcs (fully or partially) coincide.

Let  $\Gamma_D = \bigcup_{i \in \mathcal{D}} \Gamma_i$  and  $\Gamma_N = \Gamma - \Gamma_D$  be the Dirichlet and the Neumann boundary, respectively. We shall be interested in solving the problem

$$-\Delta u + u = f \text{ on } \Omega \quad (3.1a)$$

$$u = h \text{ on } \Gamma_D \quad (3.1b)$$

$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma_N. \quad (3.1c)$$

We will cast the problem (3.1) (for  $h = 0$ ) into the form of a  $\{B, F\}$  problem. To this end let  $H = H_1 = H_2 = H_D^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_D\}$  where by  $H^1(\Omega)$  we denoted the usual Sobolev space of functions with the square integrable first derivatives. Let

$$B(u, v) = \int_{\Omega} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + uv \right) dx dy \quad (3.2a)$$

and

$$F(v) = \int_{\Omega} f v dx dy + \int_{\Gamma_N} g v ds. \quad (3.2b)$$

If  $h \neq 0$ , then as usual we write  $u = z + w$  with  $w \in H^1(\Omega)$ ,  $w = h$  on  $\Gamma_D$  and  $z \in H_D^1(\Omega)$  being the solution of the  $\{B, F_1\}$  problem with  $F_1$  being properly adjusted  $F$ .

The model problem  $\{B, F\}$  satisfies the conditions listed in Section 2 provided



that  $f$ ,  $h$  and  $g$  satisfy some mild conditions as, for example,  $f \in L_2(\Omega)$ ,  $g \in L_2(\Gamma_N)$  and  $h \in H^1(\Gamma_1)$ ,  $i \in \mathcal{D}$  and  $h$  continuous on  $\Gamma_D$ .

The illustrative numerical computations presented in the next sections are related to the two dimensional elasticity problem, i.e., for the (strongly elliptic) system of two partial differential equations of second order instead of the simple model problem mentioned above. The elasticity problem has very similar property as the introduced model problem but has larger practical importance.

The finite element solution (for  $h = 0$ ) is characterized by the selection  $S_1 = S_2 = S \subset H_D^1$  and all conditions inclusive condition (2.5) in Section 2 are satisfied. If  $h \neq 0$  and  $h$  is not a trace of a function in  $S$ , then we replace  $h$  by  $h'$  which is a trace of a function in  $S$  and consider the additional error caused by this replacing.

Characterization of the solution. Set  $K$  relates to the available information about input data, i.e. the information about  $\Gamma$ ,  $f$ ,  $g$ ,  $h$ . We will assume that  $\Gamma$  is piecewise analytic,  $f$  is analytic on  $\bar{\Omega}$ ,  $g$ ,  $h$  are analytic on  $\bar{\Gamma}_1$ . This assumption, namely, that the data are piecewise analytic, is practically always satisfied in the problems of structural mechanics.

Remark 3.1. In our illustrative computations we will also present the results which are outside of the above mentioned frame, namely that  $g$  is a Dirac function (concentrated load). Such an example is well tailored for our illustrations, but needs more refined theoretical analysis which will not be addressed here.  $\Delta$

Although our main emphasis will be on the problem with piecewise analytic input data, we will also mention the results for the more usual solution set  $K$  as, for example,  $K = \{u \mid |u|_{H^k(\Omega)} \leq 1\}$ , etc.

Usual assumptions in the regularity theory of the differential equations of elliptic type are based on the theory of Sobolev spaces of finite order, i.e.  $f \in H^k(\Omega)$ ,  $g \in H^k(\Gamma)$ , etc., and often the boundary of the domain is assumed to be smooth (i.e. not piecewise smooth). Such assumptions are not sufficiently realistic in applications. Either they are too restrictive (smooth domain  $\Omega$ ) or too general ( $f \in H^k(\Omega)$ ).

To further simplify our exposition, we will assume that  $\Omega$  is a polygon. We will make some remarks about the general case.

Remark 3.2. We will also, as illustrative example, deal with one dimensional analog of our model problem, namely the problem

$$-u'' = f, \quad x \in (0,1) = I \quad (3.2a)$$

$$u(0) = u(1) = 0 \quad (3.2b)$$

with  $f$  such that the exact solution  $u_0(x)$  is

$$u_0(x) = (x-\xi)_+^\alpha + a + bx, \quad \alpha > \frac{1}{2} \quad (3.3)$$

where

$$(x-\xi)_+ = \begin{cases} x-\xi & \text{for } x \geq \xi \\ 0 & \text{for } x < \xi \end{cases}$$

and  $a, b$  are such that (3.2b) is satisfied. Obviously  $u_0 \in H^1(I)$ .  $\Delta$

#### 4. CHARACTERIZATION OF THE SOLUTION SET

As it was said earlier, the solution set  $K$  describes the solutions of the class of problem to be solved. The performance of the method is then directly related to this set.

Let  $\underline{\beta} = (\beta_1, \dots, \beta_n)$  be a  $n$ -tuple of real numbers  $0 < \beta_i < 1$ ,  $1 \leq i \leq n$ . For any integer  $k > 0$  we shall write  $\underline{\beta} + k = (\beta_1 + k, \beta_2 + k, \dots, \beta_n + k)$ . By  $r_i(x)$ ,  $i = 1, \dots, n$  we shall denote the Euclidean distance between  $x \in \Omega$  and the points  $B_i$ ,  $i = 1, \dots, n$ . We denote then  $\Phi_{\underline{\beta}+k}(x) = \prod_{i=1}^n r_i^{\beta_i+k}(x)$  and  $\Phi_{\underline{\beta}+k}(x) = \prod_{i=1}^n r_i^{\beta_i+k}(x)$ . The points  $B_i$  could be located at the boundary of  $\Omega$ , e.g. in the vertices  $A_i$  or outside. They also can be absent, but we will not elaborate on this case in this paper.

Define now

$$K_1 = \{u \in H_D^1(\Omega) \mid \left( \int_{\Omega} |D^\alpha u|^2 \Phi_{\underline{\beta}+k-2}(x) dx dy \right)^{1/2} \leq C d^{k-2} (k-2)!,$$

$$k = 2, 3, \dots, |\alpha| = k, \quad d > 1, \quad d \text{ independent of } k\}.$$

As usual, we denoted  $\alpha = (\alpha_1, \alpha_2)$ .  $|\alpha| = \alpha_1 + \alpha_2$ ,  $\alpha_i \geq 0$ ,  $i = 1, 2$ , integers and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}.$$

The functions belonging to  $K_1$  are analytic on  $\bar{\Omega} - \sum_{i=1}^n B_i$ . If  $B_i \in \partial\Omega$ , then they have singular behavior in the neighborhood of  $B_i$ , and the character of the singularity is given by  $\beta_i$  and  $d$ .

It has been shown in Babuška, Guo [1986] that if the domain  $\Omega$  is a polygon,  $B_i = A_i$  (i.e.  $B_i$  are the vertices) and functions  $f, g, h$  are analytic on  $\bar{\Omega}$  and  $\bar{\Gamma}_1$ , respectively, then the solution of the problem (3.1) belongs to  $K_1$  for a properly chosen constants  $\beta, C, d$ . The case  $B_i \notin \bar{\Omega}$  characterizes the solutions with the singularities outside of  $\Omega$ , e.g., when the domain has circular arcs and  $h = 0$ . This case describes also well the case when the natural domain of the analyticity of the solution contains  $\bar{\Omega}$ .

The set  $K_1$  obviously belongs to the family of countably normed spaces. For more about this family, we refer to Gelfand, Shilov [1964].

Let us now introduce the more standard family of solutions sets

$$K_2 = \{u \in H_D^1(\Omega) \mid \|u\|_{H^1(\Omega)}^{k_1} \leq C, \quad k_1 > 1\}$$

$$K_3 = \{u \in H^1(\Omega) \mid \|u\|_{H^2(\Omega)}^{k_2} \leq C, \quad k_2 > \frac{3}{2}\}$$

$$K_4 = \{u \in H_D^1(\Omega) \mid u = r_i^{\alpha_i} |\log r_i|^{\gamma_i} \varphi_i(\theta_i) \chi_i(r_i)\}$$

where  $(r_i, \theta_i)$  are the polar coordinates with the origin in the vertex  $A_i$ ,  $\alpha_i > 0$  noninteger,  $\varphi_i(\theta_i)$  is a  $C^\infty$  cut off function.

The motivation of the solution set  $K_j$ ,  $j = 2, 3, 4$  is that the solution  $u$  of (3.1) can be written in the form

$$u = u_1 + u_2 + u_3 \quad (4.1)$$

where

$$u_1 \in K_2, \quad u_2 \in K_3$$

and

$$u_3 = \sum C_{i,j} u_i^{[j]}, \quad u_i^{[j]} \in K_4.$$

Functions  $u_1$  and  $u_3$  satisfy the homogenous Dirichlet (essential) conditions, while  $u_2$  relates to the nonhomogeneous Dirichlet conditions. The restriction  $k_2 > 3/2$  has been made for simplicity only and can be replaced by  $k_3 > 1$ . For the theory leading to the form (4.1) we refer to Kondrat'ev [1967], Kondrat'ev, Olejnik [1983] and Grisvard [1985].

Remark 4.1. We restricted ourself to the problem (3.1) only. The practically important case of nonhomogeneous materials which is described by the equation

$$\frac{\partial}{\partial x} a \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} a \frac{\partial u}{\partial y} = f$$

with  $a$  being a piecewise constant on the domains bounded by the piecewise analytic curves can be handled in a similar way. Analogous situation occurs also when dealing with the problem of elasticity.  $\Delta$

## 5. THE FINITE ELEMENT SPACE $S$

We introduce now the finite element spaces we will deal with later.

For reasons of the simplicity of the exposition, we will restrict our choices, but our numerical example will also present more general cases.

Let  $M = \{T\}$  be a family of meshes  $T = \{\tau_i\}$  where  $\tau_i \subset \Omega$  is an open triangle, called element. Let  $h_{\tau_i} = \text{diam } \tau_i$ ,  $h(\tau) = \max_{\tau_i \in T} h_{\tau_i}$ , and let  $\rho_{\tau_i}$  be the diameter of the largest ball contained in  $\tau_i$ . We will assume that  $M$  is such that for any  $\tau_i \in T \in M$

$$\frac{h_{\tau_i}}{\rho_{\tau_i}} \leq \beta \quad (5.1)$$

i.e. that all triangles satisfy the minimal angle condition. Further we shall assume that  $\bar{\Omega} = \bigcup_{\tau_i \in T} \bar{\tau}_i$  and that any pair  $\bar{\tau}_i, \bar{\tau}_j \in T, i \neq j$  has either an entire side or a vertex in common, or their intersection is empty. The number of elements of  $T$  will be called cardinality of  $T$  and denoted by  $M(T)$ .

Remark 5.1. We restricted ourselves to triangles only. The results we will present are valid more general, e.g. for rectangles, curvilinear, triangles and rectangles.  $\Delta$

In addition to a general mesh, we will introduce two special families of meshes, the quasiuniform and the geometric mesh.

a) The family  $M_U^Y, 1 \leq Y < \infty$ , of quasiuniform meshes: There exists constant  $Y$  such that

$$h(\tau) = \max_{\tau_i} h_{\tau_i} \leq Y \min_{\tau_i} h_{\tau_i}$$

holds for any  $\tau \in M_U^Y$ .

b) The family  $M_G^{q,k}(B_1, \dots, B_n), B_j \notin \bar{\Omega}, j = 1, \dots, n, 0 < q < 1, 1 \leq k < \infty$  of geometric meshes:

Let  $\tau_i \in M_G^{q,k}, \bar{\tau}_i \cap \bigcup_{j=1}^n B_j = \emptyset$ , then there exists

$$\min_{1 \leq j \leq n} d(B_j, \tau_i) = \kappa(i)$$

and

$$k^{-1} \frac{1-q}{q} \leq \frac{\kappa(i)}{h(\tau_i)} \leq \frac{1-q}{q} k$$

where by  $d(B_j, \tau_i)$  we denoted the euclidean distance between  $B_j$  and  $\tau_i$ .

Obviously, if  $\tau \in M_U^1$ , then  $M(T) = h^{-2}(T)$  and the cardinality  $M(\tau)$  can be arbitrarily large. If  $B_j \notin \bar{\Omega}, j = 1, \dots, n$ , then for any  $M_G^{q,k}$  we have a  $M(T) \leq Q$  where  $Q$  depends on  $\Omega, \beta, q, k$ . If  $B_j \in \partial\Omega$ , then the cardinality can be arbitrarily large.

Finally we denote  $S(T, p) = \{u \in H_D^1(\Omega) \mid u|_{\tau_i} \text{ is a polynomial of degree } \leq p \text{ for any } \tau_i \in T\}$ . By  $N(S, T, p) = \dim S(T, p)$  we denote the number of degrees of freedom.

If  $T \in M_U^Y$ , then  $N(S, , p) = p^2 h^{-2}(\cdot)$ . If  $\tau \in M_G^{q,k}, B_i \notin \bar{\Omega}$ , then  $N(S, T, p) = 2$  and if  $T \in M_G^{q,k}$  with  $B_i \in \partial\Omega$ , then  $N(S, , p) = p^2 |\log_{\min}(T)|$ .

Remark 5.2. We assumed only triangular elements. For the rectangular elements, instead of polynomials of the total degree  $p$ , we use polynomials of degree  $p$  in every variable. In the case of the curvilinear elements we use the standard mapped polynomials.  $\Delta$

Remark 5.3. We mentioned only the case when the degree  $p$  is uniform, i.e. is the same for all elements. The available theory covers also the nonuniform, selective choice of the degrees  $p$ .  $\Delta$

Remark 5.4. The meaning of the finite element subspaces is especially clear in the one dimensional setting. For example, in the case of the geometric mesh with  $A = 0$ , then nodal  $x_i$  points are

$$x_i = q^{M-i}, \quad i = 0, 1, \dots, M$$

where  $q < 1$  is called the ratio of the mesh.  $\Delta$

## 6: THE BASIC APPROXIMATION THEOREMS

We will mention here some basic approximation for  $H_1 = H^1(\Omega)$ , i.e. we will study the magnitude of  $Z(K_j, H^1(\Omega), S(T, p))$ . Numerical illustration will be given in Section 7.

First let us consider any fixed mesh  $T_0$  and consider the  $p$ -version.

Theorem 1.

$$Z(K_2, H_D^1(\Omega), S(T_0, p)) \leq C p^{-(k_1-1)} = C N^{-\frac{k_1-1}{2}} \quad (6.1)$$

holds for any mesh  $T_0$  (satisfying (5.1)). For the proof, see Babuška, Suri [1985].

Theorem 2.

$$Z(K_4, H_D^1(\Omega), S(T_0, p)) \leq C |\log p| p^{-2\alpha} = C |\log N| N^{-\alpha}. \quad (6.2)$$

For the proof, see Babuška, Suri [1985].

Theorem 3. Let  $K = K_1$  and  $B_i \notin \bar{\Omega}$ ,  $i = 1, \dots, n$ . Then

$$Z(K_1, H_D^1(\Omega), S(T_0, p)) \leq C e^{-\alpha p} = C e^{-\alpha N^{1/2}} \quad (6.3)$$

where  $\alpha$  depends on  $B_i$ ,  $i = 1, \dots, n$  but is independent of  $p$ . For the proof, see Guo, Babuška [1986].

So far we assumed that the essential boundary conditions are homogeneous. Let us discuss now the case  $K_3$  when the Dirichlet conditions are not homogeneous. Let  $\tilde{\Gamma} \subset \Gamma_D$  be any side of an element  $\tau \in T$  with the end points  $P_i$ ,  $i = 1, 2$  and let  $\varphi$  be defined on  $\tilde{\Gamma}$  so that

- a)  $\varphi$  is a polynomial of degree  $p$
- b)  $\varphi(P_i) = u(P_i)$ ,  $i = 1, 2$ ,  $u \in K_3$  (because  $u \in K_3$ ,  $u \in H^{k_2}(\Omega)$ ,  $k_2 > 3/2$ ,  $u(P_i)$  is well defined)

- c)  $\int_{\tilde{\Gamma}} \varphi' \psi' ds = \int_{\tilde{\Gamma}} u' \psi' ds$  for any  $\psi$  being polynomial of degree  $p$  and  $\psi(P_i) = 0$ ,  $i = 1, 2$ .

Now we replace the boundary condition on  $\Gamma_D$  by  $\varphi$ , i.e.  $u_S = \varphi$  and then we get

Theorem 4.

$$Z(K_3, H^1(\Omega), S(\tau_0, p)) \leq C p^{-(k_2-1)} = C N^{-\frac{k_2-1}{2}}. \quad (6.4)$$

For the proof, see Babuška, Suri [1985].

Remark 6.1. In the one dimensional case we can prove more exact theorems. Let us consider the problem (3.2) with the solution (3.3) and the case of one element only. Then we have

Theorem 5. Let  $\hat{H}_0^1(I) = \{u' \in L_2(I), u(0) = u(1) = 0\}$  and  $\|u\|_{\hat{H}_0^1(I)} = \|u'\|_{L_2(I)}$ . Then

a) if  $\xi = 0$  then

$$Z(u_0, \hat{H}_0^1(I), S(p)) = C_0(\alpha) \frac{1}{p^{2\alpha-1}} (1 + O(\frac{1}{p})), \quad [p \rightarrow \infty] \quad (6.5a)$$

with

$$C_0(\alpha) = \frac{\alpha \Gamma(\alpha)^2 |\sin \pi \alpha|}{\pi \sqrt{2\alpha-1}}, \quad (6.5b)$$

b) if  $\psi < 0$ , then

$$Z(u_0, \hat{H}_0^1(I), S(p)) = C_1(\alpha) \left(\frac{1-r^2}{2r}\right)^{\alpha-1} \frac{r^p}{p^\alpha} (1 + O(\frac{1}{p})) \quad (6.6a)$$

where

$$\alpha > 0, \quad r = \frac{\sqrt{1-\xi} - \sqrt{-\xi}}{\sqrt{1-\xi} + \sqrt{-\xi}}, \quad C_1(\alpha) = \frac{\alpha \Gamma(\alpha) |\sin \pi \alpha|}{\sqrt{\pi} 2^{\alpha-1/2}}, \quad (6.6b)$$

the form  $O(\frac{1}{p})$  is uniform with respect to  $\xi \leq -\epsilon$ ,  $\epsilon > 0$  and we have the estimation: if  $0 < r^2 \leq 1 - \frac{1}{p}$ , then

$$Z(u_0, \hat{H}_0^1(I), S(p)) = \frac{1}{\sqrt{1-r^2}} \frac{r^{p+1-\alpha}}{p^\alpha} \left[ \frac{1}{p^{\alpha-1/2}} + (1-r^2)^{\alpha-1/2} \right] \quad (6.6c)$$

if  $1 - \frac{1}{p} r^2 < 1$ , then

$$Z(u_0, \hat{H}_0^1(I), S(p)) = \frac{r^{p+1-\alpha}}{p^{\alpha-1/2}} \left[ \frac{1}{p^{\alpha-1/2}} + (1-r)^{\alpha-1/2} \right] \quad (6.6d)$$

where equivalency constants depend only on  $\alpha$ ,

c) if  $0 < \xi < 1$ , then there exists a constant  $C > 0$  depending only on  $\alpha$  such that

$$Z(u_0, \hat{H}_0^1(I), S(p)) \leq \begin{cases} C(\frac{\delta}{p})^{\alpha-1/2}, & 0 < \theta < \pi - \frac{1}{p} \\ C(\frac{1}{p})^{2\alpha-1}, & -\frac{1}{p} \leq \theta < \pi \end{cases} \quad (6.6e)$$

where  $\delta = \min(\theta, r-\theta)$ ,  $\theta = \arccos(1\xi-1)$ . On the other hand, if  $\varepsilon \leq 0 \leq \pi - \varepsilon$ ,  $\varepsilon > 0$ , then there is a constant  $C(\alpha, \varepsilon) > 0$  such that

$$Z(u_0, \hat{H}_0^2(I), S(p)) \geq C(\frac{\theta}{p})^{\alpha-1/2}. \quad (6.6f)$$

For the proof, see Gui, Babuška [1986].

If the singularity is outside the domain, the accuracy increases dramatically with  $p$ . In Fig. 6.1 we see the value of  $Z$  for  $\alpha = 1.5$  in dependence on  $\xi$  and  $p$ .

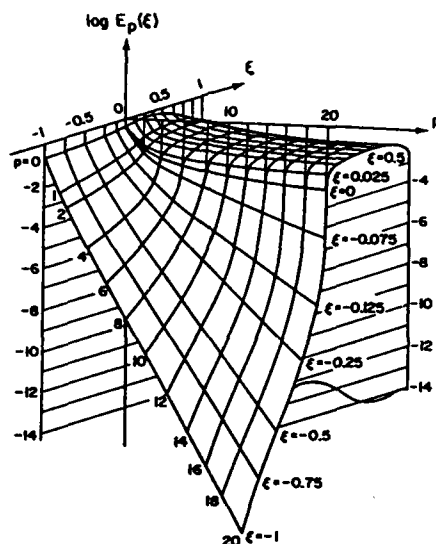


Fig. 6.1. The error of the  $p$ -version with one element. Δ

Let us now consider the  $h$ - $p$  version on the quasiuniform mesh  $\tau$ .

Theorem 6. Let  $K = K_2$ , respectively  $K_3$ . Then

$$Z(K, H_0^1(\Omega) \text{ (resp. } H^1(\Omega)), S(T, p)) \leq C(h(\tau)) h^{\eta_i-1} (\tau)^p \frac{(k_i-1)}{p} = CN^{-\frac{\eta_i-1}{2} - (k_i-\eta_i)} \quad (6.7)$$

$$\eta_i = \min(p+1, k_i), \quad i = 2, 3.$$

The constant  $C$  in (6.7) is independent of  $h$  and  $p$ .

For the proof, see Babuška, Suri [1986a].

The estimate for  $K_3$  has to be understood in the same way as in Theorem 4.

Theorem 7. Let  $K = K_4$ . Then

$$Z(K_4, H_D^1(\Omega), S(T, p)) \leq Cg(h, p, \gamma) \min(h^\alpha, h^{\min(\alpha, p-\alpha)}) p^{-2\alpha}, \quad (6.8)$$

where

$$g(h, p, \gamma) = \max(|\log h|^\gamma, |\log p|^\gamma).$$

For the proof, see Babuška, Suri [1968a].

Finally, let us address the case of the h-p version on the geometric mesh.

Theorem 8. Let  $K = K_1$  and let  $B_i$  be vertices of  $\Omega$ . Then there exists a mesh and the degree  $p$  (dependent on the mesh) such that

$$Z(K_1, H_D^1(\Omega), S(\tau, p)) \leq C e^{-\alpha \sqrt[3]{N}} \quad (6.9)$$

where  $\alpha$  is independent of  $N$ .

For the proof, see Guo, Babuška [1986].

Remark 6.2. If the degrees of the elements are uniform, then the class  $K_1$  can be extended. See, Guo, Babuška [1986c]. Δ

Remark 6.3. In one dimensional setting much more can be said. We will discuss the case  $\xi = 0$ .

Let us first ask the question about the lower bound of the error  $Z(u_0, H_0^1(I), S(\tau, p))$  among all meshes and all degree distributions. The answer is given in

Theorem 9.

$$Z(u_0, H_0^1(I), S(\tau, p)) \geq C(\alpha) \frac{1}{\sqrt{N}^{\alpha-1/2}} q_0^{\sqrt{(\alpha-1/2)}N} \quad (6.10)$$

where

$$q_0 = (\sqrt{2} - 1)^2.$$

For the proof, see Guí, Babuška [1986].

Let us now consider the geometric mesh with the ratio  $q$  and various degree distribution. We have

Theorem 10. As  $N \rightarrow \infty$  the optimal degree distribution tends to be linear with the slope

$$s_0 = (\alpha - 1/2) \frac{\ln q}{\ln r}, \quad (6.11a)$$

with

$$r = \frac{1 - \sqrt{q}}{1 + \sqrt{q}}. \quad (6.11b)$$

This means precisely that for each  $i = 1, 2, \dots$

$$\lim [p_{M_N-1}^{(M_N)} - p_{M_N-i-1}^{(M_N)}] = s_0.$$

For the proof, see Guí, Babuška [1986].

For the error estimates with linear degree vector of slopes  $S$  we have the following

Theorem 11. For the geometric mesh with the ratio  $q$  combined with a linear



degree vector of the slopes  $s$  we have

a) if  $s > s_0$ , then

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha, q, s) q^{(\alpha - \frac{1}{2})\sqrt{2N/s}} \quad (6.12a)$$

b) if  $s < s_0$ , then

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha, q, s) r^{\sqrt{2Ns}} \quad (6.12b)$$

c) if  $s = s_0$ , then

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha, q) e^{-\sqrt{(\alpha - \frac{1}{2})N} \sqrt{2 \ln q \ln r}} \quad (6.12c)$$

where  $r = \frac{1-\sqrt{q}}{1+\sqrt{q}}$  and  $s_0 = (\alpha - \frac{1}{2}) \frac{\ln q}{\ln r}$  is the optimal slope in the sense that the exponential rate attains maximum. Furthermore, the optimal geometric mesh and linear degree vector combination is given by

$$\begin{aligned} q_{op} &= (\sqrt{2} - 1)^2 \\ s_{op} &= 2\alpha - 1. \end{aligned} \quad (6.13)$$

In this case

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha) [(\sqrt{2} - 1)^2]^{\sqrt{(\alpha - \frac{1}{2})N}} \quad (6.14)$$

In (6.12) - (6.16) the equivalence constants depend on  $(\alpha, q, s)$ , and  $\alpha$ , respectively.

For the proof, see Gui, Babuška [1986]. Δ

Remark 6.5. For the optimal combination of geometric mesh and linear degree vector, the estimate can be written as

$$Z(u_0, H_0^1(I), S(\tau, p)) = e^{-1.762\sqrt{(\alpha - \frac{1}{2})N}} \quad (6.15)$$

and we have seen in Theorem 9 that this exponential rate of convergence is the best possible one. Δ

Remark 6.5. We discussed in Remark 6.2 the case of linear degree vector of slopes  $s$ . Let us discuss now the case of the uniform vector of degrees.

Theorem 12. For the geometric mesh with the ratio  $q$  combined with uniformly distributed degree  $p$ , the relation between the optimal choice of  $p$  and the number of element  $M$  in the mesh is asymptotically linear, i.e.

$$p = s_0 M \quad (\text{as } M \rightarrow \infty)$$

with  $s_0$  being the same as in Theorem 10.

For the proof, see Gui, Babuška [1986].

The analog of Theorem 11 is:

Theorem 13. For the geometric mesh with ratio  $q$  and the uniformly distributed degree  $p$  related with the number of elements  $M$  by  $p = sM$ , we have

a) if  $s > s_0$ , then

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha, q, s) \frac{q^{(\alpha-1/2)\sqrt{N/s}}}{(\sqrt{sN})^{2\alpha-1}} \quad (6.16a)$$

b) if  $s < s_0$ , then

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha, q) \frac{r^{\sqrt{sN}}}{(\sqrt{sN})^\alpha} \quad (6.16b)$$

c) if  $s = s_0$ , then one gets optimal rate of convergence

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha, q) \frac{e^{-\sqrt{(\alpha-1/2)N} \sqrt{\ln r \ln q}}}{(\sqrt{N})^\alpha} \quad (6.16c)$$

where

$$s_0 = \frac{(\alpha-1/2)\ln q}{\ln r}, \quad r = \frac{1-\sqrt{q}}{1+q}, \quad \alpha = \min(2\alpha-1, \alpha). \quad (6.16d)$$

The optimal combination is also given by

$$q = q_{op} = (\sqrt{2} - 1)^2 \quad (6.16e)$$

$$s = s_{op} = 2\alpha - 1$$

and for optimal combination we get

$$Z(u_0, H_0^1(I), S(T, p)) = C(\alpha) \frac{[(\sqrt{2}-1)^2]^{\sqrt{(\alpha-1/2)N/2}}}{(\sqrt{N})^s}. \quad (6.16f)$$

For the proof, see Gui, Babuska [1986].  $\Delta$

Remark 6.6. We can also interpret the optimal h-p version with uniform  $p$  as the envelope of the h version with fixed  $p$ . In this case the for  $N \rightarrow \infty$  the mesh tends to be geometric with the ratio  $q = e^{-4/e^2} = 0.5820$  and the relation between the degree  $p$  and the number of elements  $M$  tends to be linear with  $p = (4/e^2)(\alpha-1/2)M = 0.5413(\alpha-1/2)M$ . For more detailed analysis, we refer once more to Gui, Babuska [1986].  $\Delta$

Remark 6.7. In the case  $K = K_1$  Theorem 8 holds when we restrict ourself to the nonuniform distribution of degrees. The result holds also if curvilinear elements which have to satisfy certain conditions.

In the case when only uniform distribution of degrees is used, then these conditions are weaker than in the general case. See Guo, Babuška [1986c].  $\Delta$

Remark 6.8. We discussed only the model problem (3.1). The results hold for more general equations. For the higher order equations, see Guo, Babuška [1986b], and Babuška, Suri [1986c]. In the case of systems of second order or higher order differential equations, there is much broader scope of essential boundary conditions. For their treatment we refer to Babuska, Suri [1986b].  $\Delta$

Let us now make some comments to the theorems we mentioned above.

a) Comparing the performance of the finite element method with quasiuniform mesh with respect to the number of degrees of freedom, then the p-version (with few elements) perform better than the h-version. In the case that the solution has singularity of the type occurring in the corner of the domain, the rate of the p-version is twice that of the h-version. For the h-version there is a classical theorem mentioned in the basic books about finite element method

$$\inf_{w \in S(\tau, p)} \|u - w\|_{H^1(\Omega)} \leq C(p) h^\eta \|u\|_{H^k(\Omega)} \quad (6.17a)$$

$$\eta = \min(p, k-1) \quad (6.17b)$$

where  $C(p)$  is not specified (more precisely, the proof indicate that  $C(p) \rightarrow \infty$  as  $p \rightarrow \infty$ ). This leads sometimes to the (false) statement that (6.17b) indicate that for singular solution it is improper to use higher order elements.

b) The h-p version leads to the exponential rate of convergence when the input data are piecewise analytic. In the one dimensional case, the ratio  $q = (\sqrt{2} - 1)^2$  is the optimal one independently of the strength of the singularity. In the two dimensional case, the ratio of the same magnitude seems to be optimal although detailed theoretical analysis is not available yet. Some numerical evidence will be presented below. For practical reasons  $q = 0.15$  is recommended.

c) Although we mentioned only the simple model problem, the results hold much more generally. We mention especially the elasticity problem.

d) Very important problems arise in relation with the "locking" problem as in the elasticity problems with near incompressibility (Poisson ratio  $\nu = 1/2$ ). It has been shown by Vogelius [1983] that the p (and h-p version) is not influenced by the locking problem and solves reliably the elasticity problems with nearly incompressible materials without any difficulties. See also Babuska, Szabo [1982].

## 7. NUMERICAL ILLUSTRATIONS

In this section we will present numerical illustrations related to the theorems we mentioned in the previous section.

Example 7.1. Let us consider the plane strain elasticity problem when  $\Omega$  is the L-shaped domain shown in Fig. 7.1.

Let us assume that on  $\partial\Omega$  tractions are prescribed, i.e.  $\Gamma_D = \emptyset$ . We will assume that the solution of this problem is the displacement vector  $u = (u_1, u_2)$  where

$$u_1 = \frac{1}{2G} r^\alpha [(\kappa - Q(\alpha + 1)) \cos \alpha \theta - \alpha \cos(\alpha - 2)\theta] \quad (7.1)$$

$$u_2 = \frac{1}{2G} r^\alpha [(\kappa + Q(\alpha + 1)) \sin \alpha \theta - \alpha \cos(\alpha - 2)\theta]$$

where

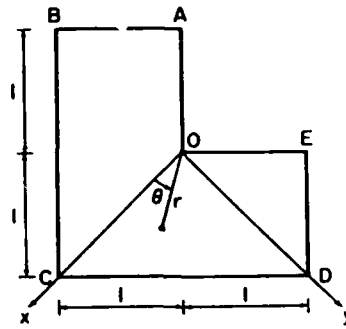


Fig. 7.1. L-shaped domain.

$$\alpha = 0.544483737$$

$$Q = 0.543075579.$$

$G$  is the modulus of rigidity and  $\kappa = 3-4\nu$ , where  $\nu = .3$ . The sides  $OA$  and  $OE$  are traction free. The solution has a typical singularity at  $O$  and is the first mode of the stress intensity factor solution. Instead of the norm  $|\cdot|_{H^1(\Omega)}$  we will be interested in the energy norm  $|\cdot|_E$  which is equivalent to the norm  $|\cdot|_{H^1(\Omega)}$ . We then define the relative error  $|e|_{E,R} = |e|_E / |u|_E$ .

First, we will consider the case of the uniform mesh with square elements shown in Fig. 7.2.

The solution  $u$  can be interpreted as a member of the set  $K_1$ , with  $B_1 = 0$ ,  $K_2$  with  $k_2 = \alpha - \epsilon$ ,  $\epsilon > 0$  arbitrary or  $K_4$  with  $\gamma = 0$ . Interpreting the solution  $u$  as the number of the solution set  $K_4$ , the estimate (6.2) gives

$$|u - u_S| \leq C \min \left[ h^\alpha, \frac{h^{\min(\alpha, p-\alpha)}}{p^{2\alpha}} \right]$$

where  $C$  depends on  $\alpha$  but is independent of  $h$  and  $p$ .

Fig. 7.3 shows the relative error  $|e|_{E,R}$  (for different  $p$ ) in dependence on  $h$  (in  $\log |e|_{E,R} \times \log h$  scale). We also show the slope  $h^\alpha$  in the figure. We see that with respect to  $h$ , the error is in the asymptotic range also for moderate  $p$  and  $h$ . Figure 7.4 shows the error in dependence on  $p$  and different  $h$ . Because of the size of the computations the error is given for  $p > 4$  only for  $h = 1/2$ , (for  $p = 4$  and  $h = 1/10$  the number of degrees of freedom  $N = 5119$ ). The slope  $p^{-2\alpha}$  ( $2\alpha = 1.088$ ) is apparent only for  $p \geq 3$ . Figure 7.5 shows the error in dependence on the number of degrees of freedom  $N$  for various  $p$ . Also the performance of the  $p$ -version for  $h = 1/2$  is shown in Fig. 7.5. We see that the  $p$ -version is more effective than the  $h$ -version and that the theoretical asymptotic slope (shown in the figure) is achieved for moderate accuracy and  $N$ .

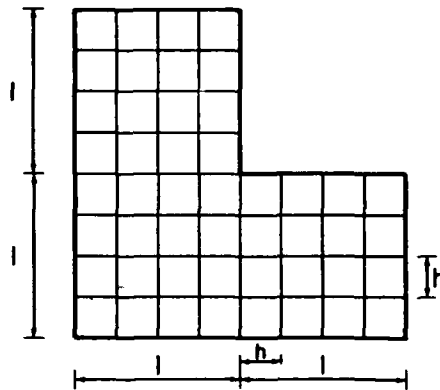


Fig. 7.2. The scheme of the uniform mesh.

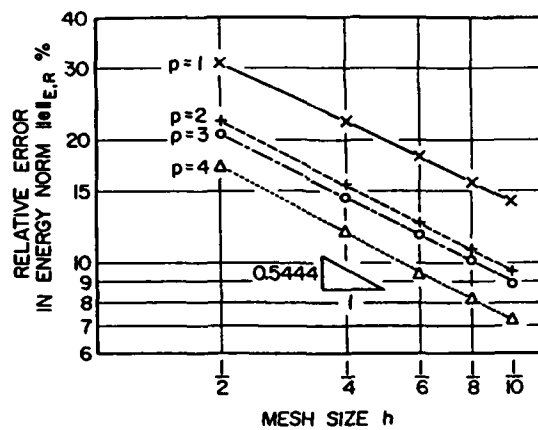


Fig. 7.3. The relative error measured in the energy norm in dependence on  $h$ .

Remark 7.2. Let us illustrate the estimate given in Theorem 5. Introduce the numerical constant  $R_p^A$

$$R_p^A = \frac{E_p}{E_p^A}$$

where  $E_p^A$  is the right hand side of (6.5) and  $E_p = \|e\|_{H_0^1(I)}$ .

Table 7.1 shows the error  $E_p$  and the numerical constant  $R_p^A$  for  $\alpha = 0.7$ , 3.5 and  $\xi = 0$ . We see that for  $\alpha$  smaller  $R_p^A \rightarrow 1$  quicker than for  $\alpha$  larger (i.e. smoother function).

For more numerical results we refer to Gui, Babuska [1986].

Δ

So far we addressed the performance of the uniform mesh. Let us discuss now the performance of the finite method on the geometric mesh. Fig. 7.6 shows the geometric mesh with  $n = 2$  layers and the ratio .15.

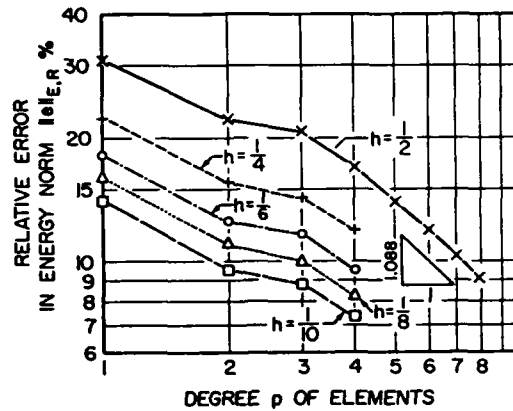


Fig. 7.4. The relative error measured in the energy norm in dependence on  $p$ .

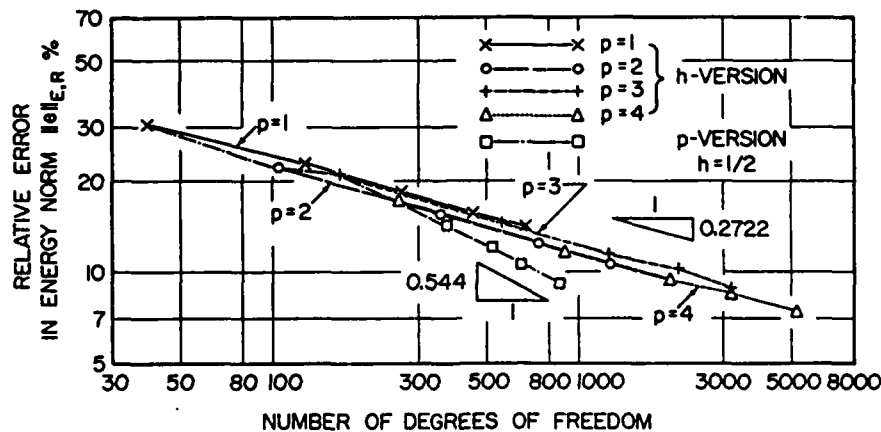


Fig. 7.5. The relative error measured in the energy norm in dependence on  $v$ .

Fig. 7.7 shows the performance of the  $p$ -version for various number  $n$  of layers in  $\log|e|_{E,R} \times \log N$  scale. Fig. 7.7 shows also for comparison the performance of the method on the uniform meshes shown in Fig. 7.2. We see that for every number of layers the rate is  $N^{-.5441}$  where  $N$  is sufficiently large (dependent on  $n$ ), and that the error has the reverse S slope form. In the first phase the accuracy is exponential (curved down) while in the second phase the accuracy is an algebraic one. Further we see that the optimal number of layers and the degree  $p$  depends on

the required accuracy. We will return to this question later in the discussion of the expert system for the h,p version. See Section 11. Fig. 7.8 shows the relative error in the  $\log|e|_{E,2} \times N^{1/3}$  scales. We see that the envelope of these curves which characterize the performance of the h-p version is nearly a straight line as it has been theoretically indicated by Theorem 8.

TABLE 7.1. The error and the values of the numerical constant in the one dimensional case.

p	$\alpha = 0.7$		$\alpha = 3.5$	
	$E_p$	$R_p^A$	$E_p$	$R_p^A$
1	4.743 E-1	0.9877	1.021	0.2032
2	3.627 E-1	0.9967	3.402 E-1	4.335
3	3.090 E-1	0.9985	3.093 E-2	4.488
4	2.756 E-1	0.9992	2.379 E-3	1.940
5	2.522 E-1	0.9995	4.760 E-4	1.480
6	2.344 E-1	0.9996	1.400 E-4	1.300
7	2.204 E-1	0.9997	5.154 E-5	1.208
8	2.090 E-1	0.9998	2.210 E-5	1.153
9	1.994 E-1	0.9998	1.057 E-5	1.118
10	1.912 E-1	0.9999	5.495 E-6	1.094
11	1.840 E-1	0.9999	3.053 E-6	1.077
12	1.777 E-1	0.9999	1.790 E-6	1.064
13	1.722 E-1	1.000	8.999 E-7	1.054
14	1.671 E-1	1.000	6.978 E-7	1.046
15	1.626 E-1	1.000	4.585 E-7	1.040

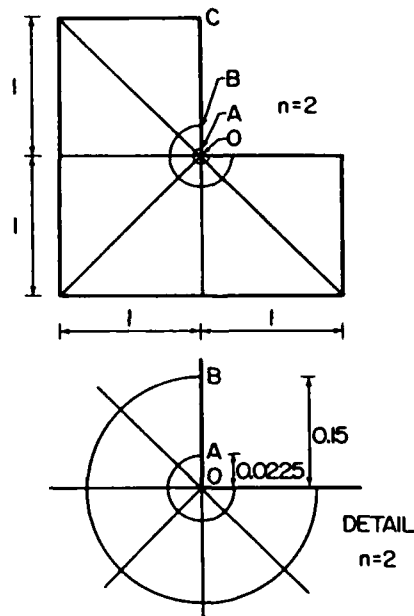


Fig. 7.6. The geometric mesh with  $n = 2$  layers.

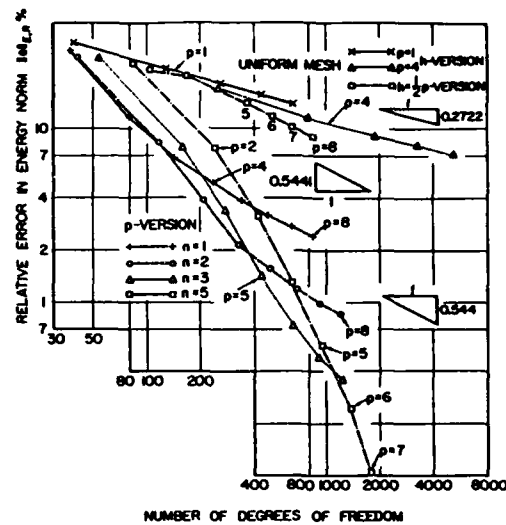


Fig. 7.7. The performance of the p-version on various meshes in the scale  $\log|e|_{E,R} \times \log N$ .

The envelope together with the optimal pair  $(n,p)$  is shown in Fig. 7.9, together with the approximate straight line.

The p and h-p version has various properties which are essentially different from the h-version. We already mentioned the robustness of the p-version with respect to the Poisson ratio (the nearly incompressible material). Let us discuss now briefly the "pollution" problem which describes the influence of the locally unsmooth solution. For the h-version the well known interior estimate shows that the influence of the local unsmoothness of the solution is also local. For the p-version the effect is not so local as for the h-version. To illustrate this effect, let us solve the plane strain elasticity problem ( $\nu = .3$ ) loaded by the concentrated load (Dirac function) as shown in Fig. 7.10. On the sides BCDC'B' the tractions are prescribed so that the exact solution is the classical Bousinesque solution on the half plane (hence B'AB side is traction free). The energy of the solution is infinite and hence the usual theory has to be generalized. We will be interested (because of the obvious symmetry) in the finite element solution and its accuracy on  $\hat{\Omega}$  (shadowed in Fig. 7.10). As before, we will measure the accuracy in the energy norm. The meshes used for the computation are shown in Fig. 7.11 (only half of the domain is shown). The domain  $\hat{\Omega}$  is once more shadowed. The minimal (pollution free) error on  $\hat{\Omega}$  is the error of the finite element method when the exact tractions are prescribed on  $\partial\hat{\Omega}$ .

Fig. 7.12 shows the error  $|e|_{E(\hat{\Omega})}$  for the meshes I-IV and the "pollution free" error. We see that the p-version for any shown mesh does not converge although for three layers the error is in the range of engineering accuracy. For



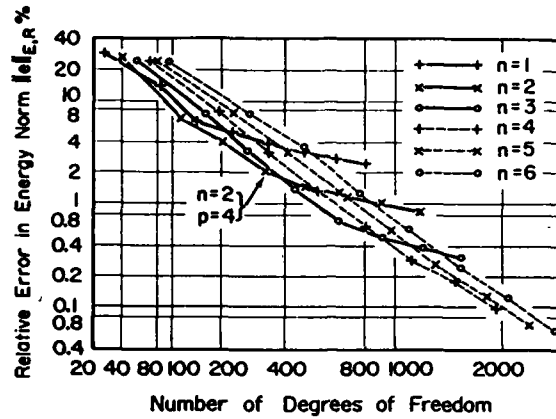


Fig. 7.8. The performance of the p-version on meshes with various number of layers  $n$  in the scale  $\log |e|_{E,R} \times N^{1/3}$ .

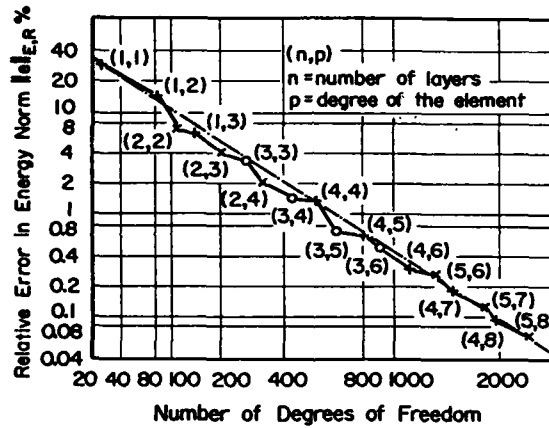


Fig. 7.9. The performance of the h-p version.

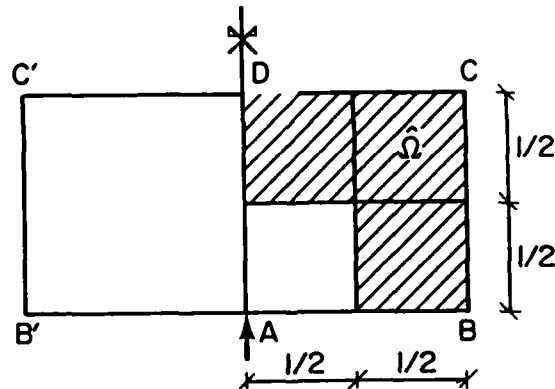


Fig. 7.10. The scheme of the problem of a concentrated load.

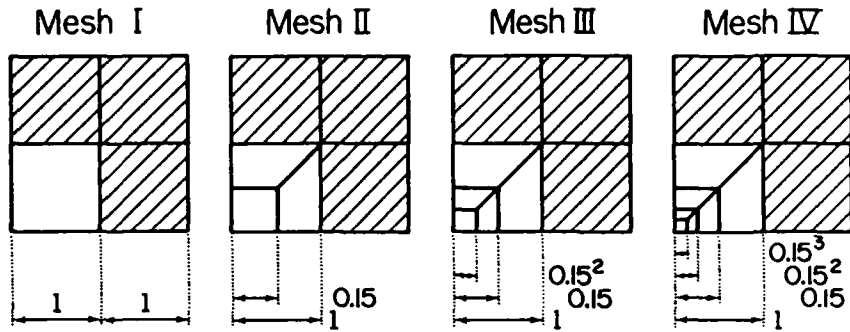


Fig. 7.11. The meshes used for the computation.

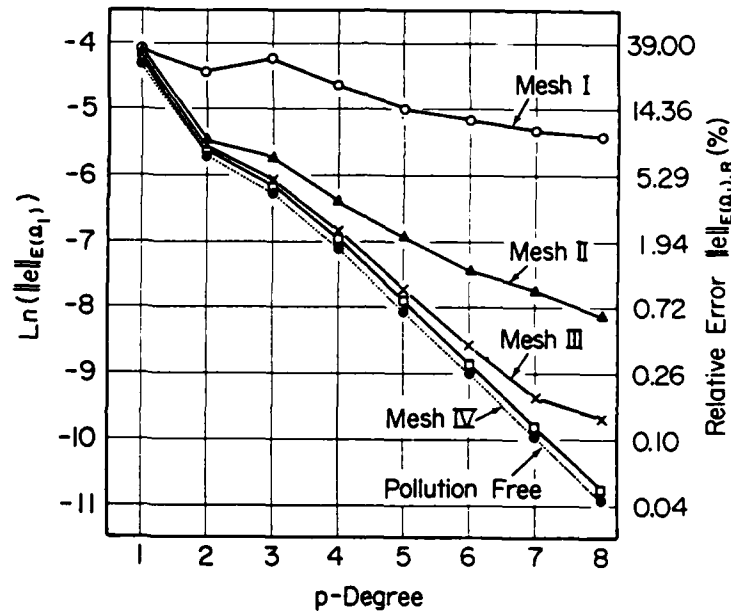


Fig. 7.12. The error  $\|e\|_{E(\hat{\Omega})}$  of the p-version and the "pollution free" error.

more details we refer to Babuška, Oh [1986]. Similar situation occurs also for weaker singularities and a high accuracy, e.g., in stresses. The accuracy has to be achieved in practice by "filtering" out the singularity influence by using few layers (in dependence on the strength of the singularity).

## 8. SOME IMPLEMENTATIONAL ASPECTS

The h and h-p versions of the finite element method give a large flexibility in the change of degrees of the elements either uniformly or element by element. This has to be respected in the architecture of the program. One of the essential features is the use of the hierarchical shape functions. In the most simple setting of a square master element shown in Fig. 8.1. The hierarchical shape functions are as follows:

We have 3 kinds of shape functions

- The modal shape functions  $\hat{N}_1(\xi, \eta)$ .

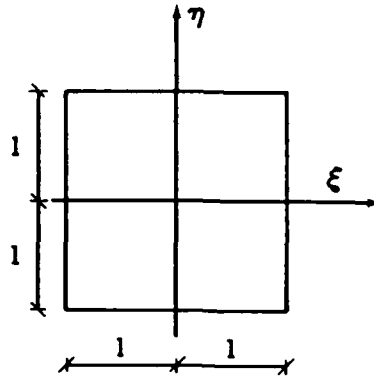


Fig. 8.1. The scheme of the master element.

- b) The side shape functions  $\overset{1}{N}_i(\xi, \eta)$ .
- c) The internal shape functions  $\overset{2}{N}_i(\xi, \eta)$ .

The index above describes the dimension of the part of the element associated with the shape functions.

a) The nodal shape functions  $\overset{0}{N}_i(\xi, \eta)$ ,  $i = 1, \dots, 4$  are the usual bilinear functions associated with the four nodes.

- b) The side shape functions  $\overset{1}{N}_i(\xi, \eta)$ ,  $i = 1, \dots, 4(p-1)$ . Let

$$P_j(\xi) = \int_{-1}^{\xi} \ell_j(x) dx$$

where  $\ell_j$ ,  $j \geq 1$  is the Legendre polynomial. The the side shape functions associated with the side  $-1 < \xi < 1$ ,  $\eta = -1$  are

$$\overset{1}{N}_j(\xi, \eta) = P_j(\xi) \frac{1-\eta}{2}, \quad j = 1, \dots, p-1$$

and hence we have  $4(p-1)$  side shape functions.

c) The internal shape functions  $\overset{2}{N}_i(\xi, \eta)$ ,  $i = 1, \dots, (p-1)^2$ . The internal shape functions are

$$\overset{2}{N}_{j,k}(\xi, \eta) = P_j(\xi) P_k(\eta) \quad j \geq 1, \quad k \geq 1.$$

The number of the internal shape functions can be reduced as in the case of the serendipity elements. In this case the total number  $Q$  of all of the shape functions is

$$Q = 4p + \max(0, \frac{1}{2}(p-2)(p-3))$$

as used in the program PROBE. See Szabó [1985]. Probe uses  $1 \leq p \leq 8$ .

The shape functions are called hierarchical because the increase of  $p$  leads only to the addition of new shape functions. The use of the integrals of Legendre

polynomials has essential advantages in the connection with the round-off error control.

The hierarchical type of this elements leads to the construction of the local stiffness matrix by bordering technique when  $p$  increases. The same structure can be achieved for the global stiffness matrix. This character then allows the flexibility to change the degree of elements and its use for the quality control and adaptive approaches.

So far we mentioned only the square elements. Similar hierarchical structure can also be used for the triangular elements.

The  $p$ -version (and  $h$ - $p$  version) utilizes large elements. Hence it is important to deal properly with the curved boundaries. It seems to be advantageous (as implemented in PROBE) that the boundary is described exactly and not to be approximated, e.g. by polynomials. This is achieved by using blending function mappings. This approach, of course, leads to the loss of the important rigid body motion property of the isoparametric elements. Nevertheless, if the boundaries are smooth, e.g. piecewise analytic, then for higher  $p$  (say  $p = 3-4$ ), the effect of this loss is in practice negligible as shown theoretically and by experience with PROBE. In addition, the quality control indicates also this possible effect (see also, Section 10).

The architecture of the  $p$ -version finite element program has essential differences in comparison to the  $h$ -version. We mention here the type of meshes which are used which influence the mesh generator and many other aspects. We will not elaborate more.

## 9. THE ASSESSMENT OF THE COMPLEXITY AND THE COMPUTATIONAL COST

The cost of using any finite element code consists of the computer cost and the human cost. The relative part of the human cost grows due to the progress in the hardware development and is often the major part of the computational analysis. Hence, the assessments of the complexity and the total cost of the method is very subtle topic. Some of the aspects will be also discussed in Section 12 where the use of the  $h$ - $p$  version and the experience with the code PROBE in the industry will be addressed.

The computer cost of running the code consists of the cost of the basic computation itself, the cost of the assessment of the quality of the results and the cost of various "bells" and "whistles" and the "niceties" for the user which any commercial code has to have. The computational cost depends also on the robustness of mesh design criterion and selection of other parameters leading to the desired accuracy, and how effectively the user is able to make the proper choice of these parameters. The computer cost is the cost of numerical operations and the IO cost which often can be the main cost. This makes any comparison a difficult task. It seems that one of the best ways is to compare various codes in an industrial environment (see also Section 12).

In general we have to relate the cost of the achieved (a-priori given by the user) accuracy and the assessment of the reliability of the computed data (see also Sections 10 and 11). In previous sections we measured the effectivity of the method by the relation between the accuracy measured in the energy norm and the number of degrees of freedom. Of course, the number of degrees of freedom does not reflect some aspects of computational effort as the computation of the local stiffness matrices, the influence of the sparsity of the global stiffness matrix and hence more detailed analysis both theoretical and experimental is in place. The only count of arithmetical operations, although important, is not sufficiently realistic basis for the computational cost. Hence, very likely the most reliable way is to write a program which implements the method and analyze its performance. To this end, we used part of the program PROBE adjusted to given purpose. Although the results depend on the programming, the computer, compiler, etc., the assessment based on such an analysis is the most reliable technique.

The major parts of the finite element computations are

- a) input, treatment of the geometry, controls of the input, etc.
- b) Computation of the local stiffness matrices and load vectors.
- c) Assembly and elimination procedure.
- d) Postprocessing inclusive accuracy assessment.

We will address here only parts b) and c). To analyze this question, let us consider the mesh shown in Fig. 9.1 with  $r \leq m$ .

We mention that this mesh is topologically very close to the mesh shown in Fig. 7.6 ( $r = \min(6, n+1)$ ,  $m = \max(6, n+1)$ ) or in Fig. 7.11. The cost of the parts b) and c) of the computations dealing with the examples treated in Section 7 (Figs. 7.5, 7.11) is essentially the same as when dealing with the mesh of Fig. 9.1 for properly chosen  $r$  and  $m$ .

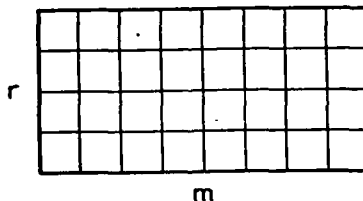


Fig. 9.1. The mesh for the complexity analysis.

Let us now count the number of operations for the two major parts b) c) mentioned above for the elasticity problem (2 unknown functions)

- a) Computation of the local stiffness matrix. It consists of
  - 1) Computation of the values of the shape function and the Jacobian in one Gaussian point.
  - 2) Computation of the values in  $\approx p^2$  Gaussian points and the sum-

mation for obtaining one of the entries of the local stiffness matrix.

3) Computations of all entries  $\approx p^4$  of the local stiffness matrix.

4) Computations of  $mn$  local stiffness matrix.

The crude operation count indicates that we can expect  $s \approx mn p^7$  operations. Nevertheless, this count is deceptive as will be seen below.

The number of the shape functions is

$$Q = [4p + \max(0, \frac{1}{2}(p-2)(p-3))]2 \quad (9.1)$$

(the serendipity type of elements are used and hence, for  $1 \leq p \leq 8$  the second term is not too large and the number of the shape functions is closer to  $\approx p$  than  $p^2$ . The number of Gaussian points used is  $2[\frac{p}{2}] + 2$  where  $[a]$  denotes the integral part of  $a$ . A careful programming leads then to  $\approx p^4$  operations in the given range. In Fig. 9.2 we show the time for computations of 25 local stiffness matrices ( $m = r = 5$ ) in dependence on  $p$  in the double logarithmic scale (time units on VAX). The straight line with the shape 4 confirms the rate  $p^4$  we expected. The minor "wabbling" of the time curve is caused by using only even number of the Gaussian points.

b) The assembly and elimination form  $m \geq r$ . The frontal solver has been used which joints the elimination and the assembly. The front is of order  $rp$  (when properly counting (condensation approach) the dealing with internal shape functions. Hence, one can expect the time is of order  $\approx (rp)^2 mrp = mr^3 p^3$ . Fig. 9.3 shows the

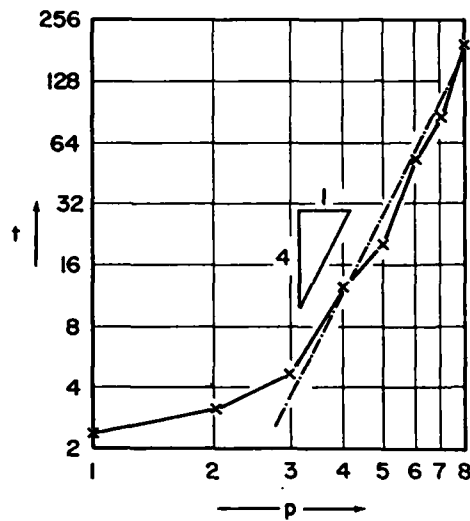


Fig. 9.2. The computer time for computation of 25 local stiffness matrices.

time in dependence on  $p$  for various values  $m$  and  $r$ . The slope  $p^3$  is depicted in the figure which agree with the expectation.

Fig. 9.4 shows the time in dependence on  $m$  with  $m = r$  and different  $p$ . The slope of  $m^3$  is shown in the figure. We see that in the given range, the rate is closer to  $m^3$  than to  $m^4$  as one could expect. For small  $p$  and  $m$  the tests are outside of the asymptotic range which had to be expected.

Analyzing Figs. 9.2-9.4 and the experience, we see that the number of operations for the  $h$ - $p$  version with number of layers roughly proportional to  $p$  (see, e.g. Fig. 7.9) leads to number of operation of order  $N^\alpha$  where  $1.5 \leq \alpha \leq 2.5$ , where with  $p$  increasing  $\alpha$  increases. The conventional  $h$ -version with  $p = 1, 2$  will practically never lead to the accuracy of order of  $1\%$  for any reasonable computational effort if the problem is not very simple. Nevertheless, for small accuracy, a small  $p$  is desirable (see Fig. 7.9). Hence, the proper design of the mesh and the degree of elements is very important. This will be discussed in the next two sections.

The method is very robust with respect to the mesh design when some basic rules are observed. The accuracy can be obtained by using higher degree without changing the mesh.

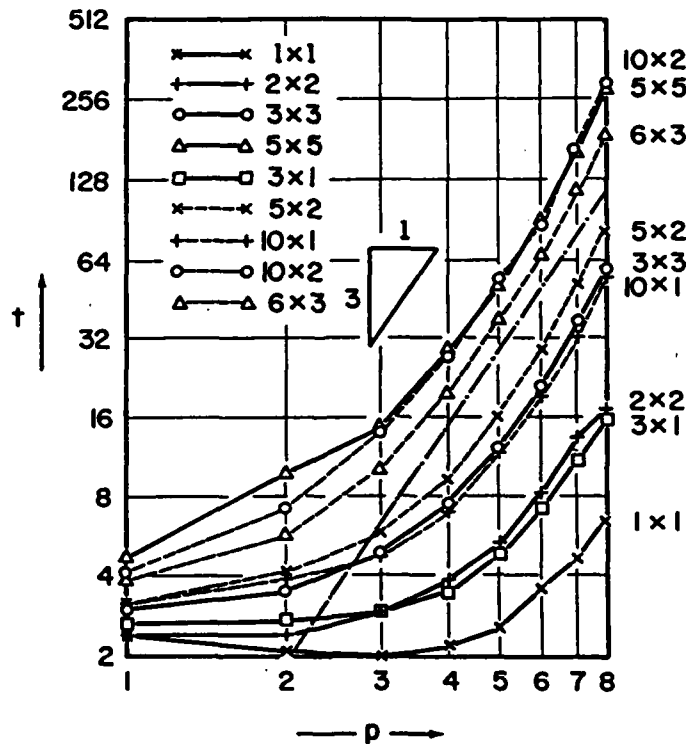


Fig. 9.3. The time of the assembly and the elimination in dependence on  $p, m, r$ .

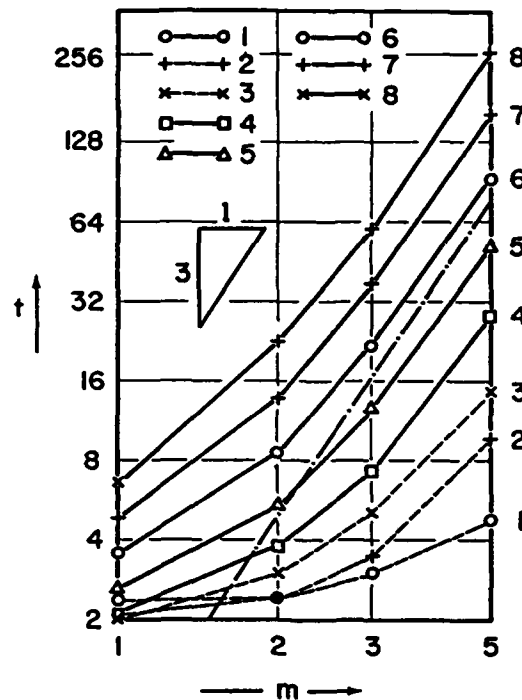


Fig. 9.4. The time of the assembly and the elimination for  $r = m$ .

#### 10. A-POSTERIORI ASSESSMENT OF THE QUALITY OF COMPUTED DATA

An essential part of the computational analysis is the a-posteriori assessment of the quality of the obtained results of interest. (For the survey of the main ideas for the quality assessment, we refer to Noor, Babuska [1986]). Although the energy norm is basic measure of accuracy, it's mostly not the most relevant measure for engineering purposes.

The basic and practically most efficient method for assessment of the accuracy of the data of interest is comparing computed data obtained for various  $p$  and use of the extrapolation method. The computed data can be of various type inclusive various equilibration checks, etc. Although PROBE provides many reliability checks, theoretically the a-posteriori quality assessment is the least mathematically understood part of the computation.

The assessment of the error measure in the energy form is the basic one and relatively easiest to obtain because  $|e|_E^2 = E_E - E_{FE}$  where  $E_E$  (respectively  $E_{FE}$ ) is the strain energy for the exact (respectively, finite element) solution and  $E_{FE}$  increases with increase of  $p$ . As we have seen earlier, the rate of convergence of the  $p$ -version with properly designed mesh is higher than the algebraic one. Hence, one can, for example (as implemented in PROBE) assume that

$$|e|_E^2 = (E_E - E_{FE}) = C(p)p^{-\beta(p)}$$



with  $C$  and  $\beta$  to be computed from 3 consecutive degrees  $p - 2$ ,  $p - 1$ ,  $p$ . Because of the reverse  $S$  property of the accuracy mentioned earlier  $\beta$  grows with  $p$  and then decreases to the limiting value when algebraic rate is achieved. The desired accuracy should be practically achieved in the range of  $p$  when  $\beta$  is about maximal. This gives an indication of the quality of the mesh for given accuracy.

We will illustrate the approach on our L-shape problem with the mesh of two layers ( $n = 2$ ) discussed in Section 7. Table 10.1 gives the basic data. Column 5 is the value of the energy norm in % of the true error. Column 7 shows the relative error in % (C.P.R.E) when computed from the current values  $p - 2$ ,  $p - 1$ ,  $p$  of the degrees. Column 8 shows the error (PRE) computed from the predicted strain energy for  $p = 6, 7, 8$ .

Coming back to Fig. 7.8 we see that the value  $p = 4 + 5$  which is indicated by  $\beta$  as the optimal one is close to the envelope of the curves. The predicted error CPRE is more reliable in the phase when  $\beta$  is increasing than decreasing which is obvious because of (10.1). Nevertheless, as a whole CPRE is very reliable.

Obviously, PRE is very accurate for lower  $p$ .

The reliability of the estimation we have shown above is closely related to the monotone behavior of the error. Nevertheless, it is necessary to estimate also other values of interest. We shall show the computation of the stress intensity factor  $K$  for the case of the L-shaped domain analyzed in Section 7. The method of computation of this stress intensity was developed in Babuška, Miller [1984] and Szabo, Babuška [1986]. It has been shown that this method gives accuracy which is roughly as the accuracy of the strain energy (i.e. the square of the energy norm error). Table 10.2 shows the values of the stress intensity factor (normalized to value 1). For more, see Szabó, Babuška [1986]. In contrast to the energy, we see that the error changes sign and is not monotonic. This, of course, complicates the extrapolation procedure.

Finally, let us show a more complicated example (where the exact solution is not known) of the cracked panel as shown in Fig. 10.1 with the used mesh.

Table 10.3 shows the values of the first and second stress intensity factor  $K_I$  and  $K_{II}$ .

The program PROBE has built in additional quality tests based on the equilibrium testing so that the user can place maximal confidence in the data of interest.

Let us underline that because of the hierarchic structure of the program the computation for different degrees can be arranged in a computationally very effective way.

As we have seen, it is important to select effectively the parameters of the  $h$ - $p$  version, i.e. the mesh and the degree of the elements which lead to the desired accuracy of the computed data of interest. This can be done by a feedback (adaptive) approach or by help of an expert system or by the combination of both.

TABLE 10.1

1	2	3	4	5	6	7	8
p	N	FE	E <sup>-</sup> FE	$\frac{ e _E}{ u _E} \%$	$\beta$	C.P.R.E.	P.R.E.
1	41	3.88608797	2.684 (-1)	25.41	--	--	25.41
2	119	4.12483255	2.971 (-2)	8.45	--	--	8.45
3	209	4.14811501	6.429 (-3)	3.93	1.93	5.34	3.91
4	335	4.15265042	1.893 (-3)	2.13	2.76	2.02	2.09
5	497	4.15362580	9.084 (-4)	1.47	3.05	1.01	1.41
6	695	4.15297464	5.695 (-4)	1.37	2.45	.80	1.10
7	929	4.15413900	4.052 (-4)	.98	1.83	.75	.89
8	1199	4.15423777	3.064 (-4)	.85	1.39	.74	.74
$\infty$	$\infty$	4.15454422					

TABLE 10.2

1	2	3
p	N	K
1	41	.95268
2	119	1.02177
3	209	1.00250
4	335	1.00073
5	497	0.99991
6	695	0.99985
7	929	0.99987
8	1199	0.99990
$\infty$	$\infty$	1.00000

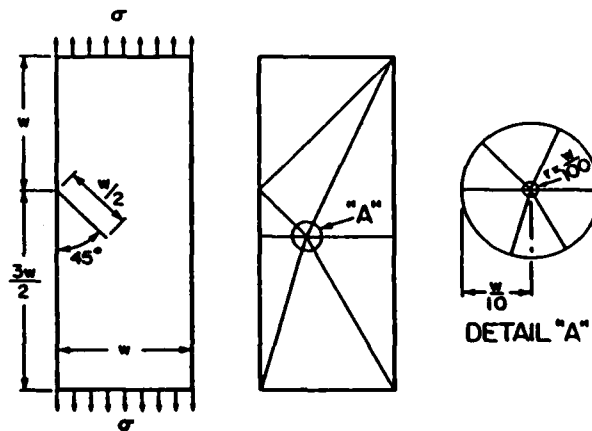


Fig. 10.1. Scheme of the cracked panel.

TABLE 10.3

1	2	3	4
p	N	$K_I$	$K_{II}$
1	43	.42259	-.37480
2	125	.55588	-.25578
3	221	.56161	-.28951
4	355	.59232	-.28319
5	527	.59825	-.29398
6	737	.60043	-.28897
7	985	.60119	-.29196
8	1271	.60132	-.29042

### 11. THE EXPERT SYSTEM AND THE FEEDBACK APPROACH

Remark 11.1. In Gui, Babuška [1986] (Part III) we developed and theoretically studied a feedback system which leads to the simultaneous selection of the mesh and (nonuniform) degrees of elements. This study has shown that the adaptive approach for the h-p version, based on similar principles as for the h version, is very complicated and other approach has to be likely considered.  $\Delta$

The feedback system for the p-version with uniform p (i.e. for fixed mesh) can be relatively easy implemented. We have seen in Sections 8 and 9 that the hierarchic structure of the p-version allows effectively change the degree. Hence, the feedback of the p-version for the given mesh consists of successive computation for increasing p until the value of the interest will have admissible accuracy. The main question is what is the ratio of the cost of the entire computation to the cost of the final one. Assuming that most ineffective computation mode, i.e. independent computation for every p starting from p = 1 and that the cost  $W(p)$  of the computation for degree p is  $W(p) = Cp^\alpha$  the ratio for  $(1 \leq p \leq 8)$  for  $\alpha = 3$  (respectively,  $\alpha = 4$ ) ranges from 1 to 2.5 (respectively, 2.14). Hence, a more effective computation arrangement exploiting the hierarchic structure leads to the ration in the range 1.5-2 at the most. This ratio is very comparable with the ratio in the adaptive approaches based on the h-version for the energy norm accuracy. The feedback p-version can be easily used for any accuracy definition.

Above we have assumed that all computations starting from p = 1 are made and that the mesh is given. The aim is obviously to construct the mesh and p = p<sub>0</sub> so that the desired accuracy is achieved and to compute only the solutions for p ≤ p<sub>0</sub> (which is very cheap because of the hierarchical structure) to get the error estimates based on the extrapolation procedure.

Recently, a lot of progress has been made in the general developing of the knowledge based systems (expert systems) in general (see, e.g. Waterman [1986], Hayes-Rogh, Waterman, Lenat [1984]) and various attempt has been made to apply the

system for computational mechanics problems (see e.g., Taig [1986], Fenves [[1986]]). In Babuška, Rank [1986] and Rank, Babuška [1986] and expert system was developed which advises the user how to use the h-p version in an effective way. One of the main problems is to develop the set of rules which is then utilized in the system.

The finite element analysis can be essentially separated into two parts, the decision phase and the execution phase, which carries out the decision although these two phases are often interwoven. The adaptive approach combines these two phases very tightly when user supplies only the basic data and the program takes care about the entire process. The approach in the expert system mentioned above, clearly separates the phases of decision and execution and adds the part of the advise and communication. It is using theoretical mathematical results mentioned in previous sections in combination with heuristics. The system was developed for treating polygonal domains and accuracy measured in the energy norm. We will show now an example. For more details, we refer to Babuška, Rank [1986].

Let the problem be the plane strain of the "cracked wrench" (symmetric) domain shown in Fig. 11.1. The first task is the construction of the geometry mesh. Fig. 11.2 shows it. From the theory of the h-p version, it is known, that areas in the neighborhood of the corners (entrant) are critical and have to be dealt with. The user is advised about it and a basic mesh which separate the critical areas is constructed. Fig. 11.3 shows this basic mesh. Program then computes rough solution on basic mesh, computes the few stress intensity factors for every corner and computes an optimization problem leading to the highest accuracy for the minimal cost, and gives for given  $p$  the (optimal) distribution of layers. Based on this computation the expert presents the user with the relation between predicted error and cost ( $N$ ) for the optimal mesh computation. Fig. 11.4 shows this graph. The user selects the accuracy and the program constructs the mesh for final solution. Fig. 11.5 shows this mesh (not all layers are shown). For example, the mesh for the error 1.8% (respectively 1.3) leads to the predicted optimal degree  $p = 4$ ,  $N = 2101$  (respectively 2517) and number of layers shown in Table 11.1. We denoted by  $SL$  (column 1) the number of the critical area as shown in Fig. 11.1 and by  $L$  the suggested number of layers in this area. The computation on this mesh leads to the errors 3.1% and 2.6%, respectively.

We see that the predicted errors are good quality although not perfect what could be expected. Nevertheless, the optimal distribution of the layers was predicted correctly.

If the error 3.1% is not acceptable, then  $p$  can be increased in the adaptive mode. In our case, this would mean to increase  $N$  to 3059 which will lead to the error 1.9%. (The other possibility is to construct the optimal mesh for higher accuracy.)

Let us mention that the expert system we outlined cost about 10-20% of the entire computer cost.

TABLE 11.1

N = 2101				N = 2517			
1	2	1	2	1	2	1	2
SN	L	SN	L	SN	L	SN	L
1	3	6	3	1	3	6	3
2	2	7	2	2	3	7	2
3	2	8	2	3	4	8	2
4	3	9	3	4	4	9	3
5	3			5	3		

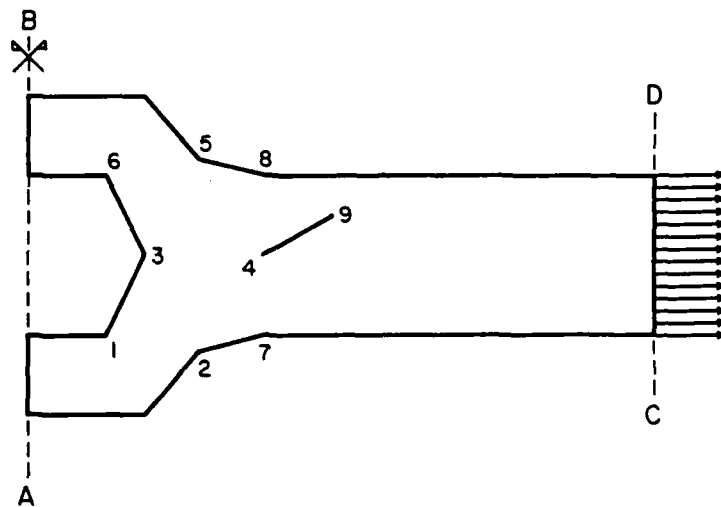


Fig. 11.1. The cracked wrench problem.

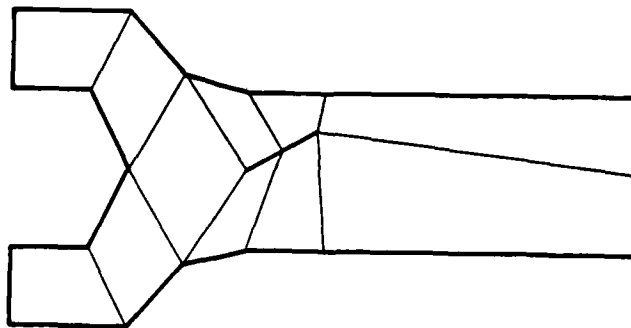


Fig. 11.2. The geometry mesh for the cracked wrench problem.

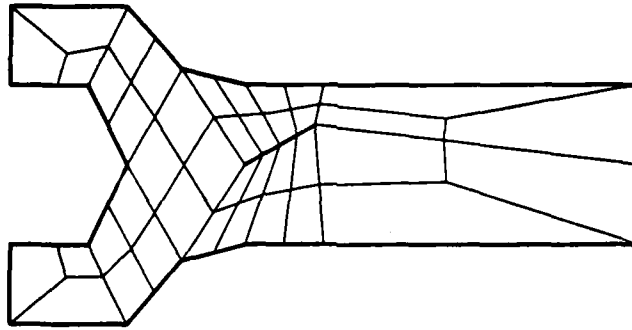


Fig. 11.3. The basic mesh for the cracked wrench problems.

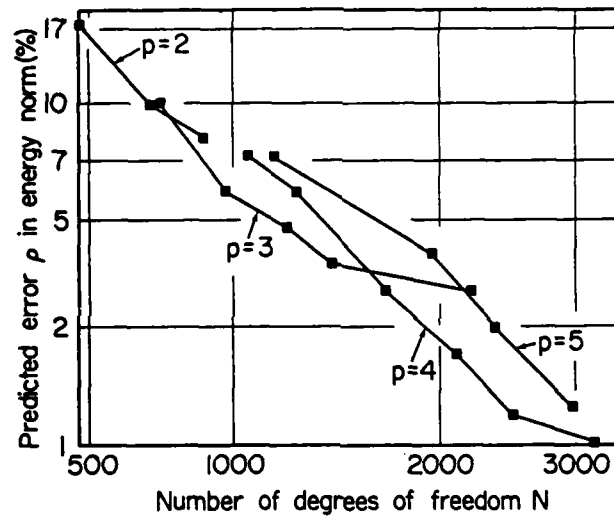


Fig. 11.4. Predicted error for cracked wrench problem.

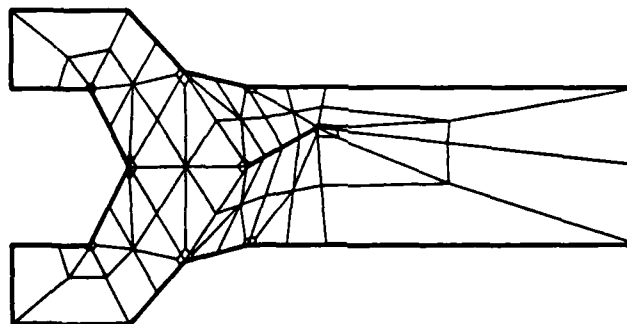


Fig. 11.5. The final mesh for cracked wrench problem.

Fig. 11.6 shows the actual error for the meshes and degrees shown in Fig. 11.4.

Finally, let us underline that the expert system and the problem of the optimal mesh design is in a very beginning stage.

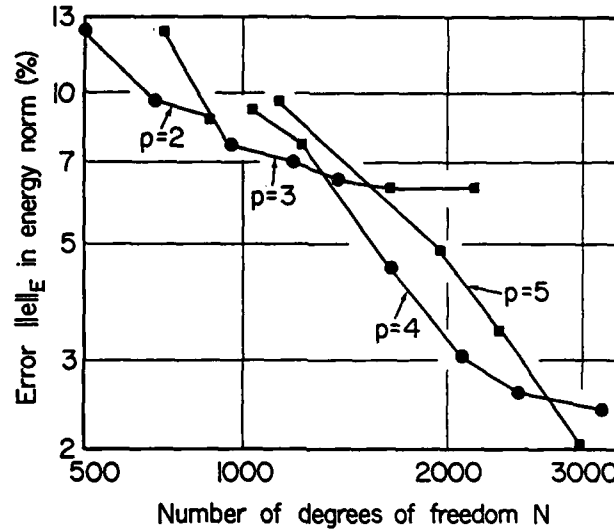


Fig. 11.6. Actual error for cracked wrench problem.

## 12. INDUSTRIAL EXPERIENCE WITH THE $p$ AND $h$ - $p$ VERSIONS OF THE FINITE ELEMENT METHOD

As was said in Section 1 the  $p$ -version of the finite element method was first introduced in USA when the program FIESTA was released - Peano, Walker [1981]. The first release of FIESTA had elastostatic capabilities for solving three dimensional problems. The  $p$ -extension (i.e., increase of  $p$ ) was possible through the specification six group of hierarchic basis function corresponding to polynomial degrees ranging from 1 to 4. The program had appealed to engineers mainly because it allowed the use of coarse meshes and therefore simplified data preparation tasks. Visual checking of the mesh in three dimensions is generally very difficult and often impossible when the  $h$ -version is used.

The industrial users are usually far more concerned about human time than machine cycle requirements but thus greatest concern is the elapsed real time between problem definition and delivery of a reliable solution. This is because the costs associated with numerical analysis are usually negligible as compared with the consequences of not being able to deliver correct solutions on time within the project schedule prescribed by management.

The first release of the program PROBE was introduced to the aerospace industry in June 1985 by Noetic Technologies Corporation, a St. Louis firm. This release had capabilities in linear plane elasticity only. The second release also has capabilities in linear two dimensional heat transfer, axisymmetric and planar thermoelasticity. The first release was of interest mainly to airframe designers, whereas the

second release is also of interest of engine designers.

The program PROBE was extensively tested and is used very successfully in the industry, e.g., in the analysis of orthotropic panels, various cut-cuts fracture mechanics, problems involving laminated composites where the peeling stress was of primary interest, etc.

Engineer users consider various features to be of important for them and judge the programs and methods accordingly. The features are as follows:

1) Increased level of confidence in the computation. The usual observation of the convergence of the data of interest (not only the energy norm) and various additional tests as equilibrium checks (having physical meaning), etc. give the user the confidence in the computational analysis. (For some comments, see Section 8.)

2) Lower human time requirement due to simplicity and flexibility of input. In an industrial test 30-40 fold saving in human time were reported in comparison with conventional finite element technology (Barhart, Eisemann [1986]).

3) To get rapid convergence and flexibility, e.g., when large aspect ration have to be used as e.g. in the case of composite joints.

4) The flexibility of mesh design, e.g. in fracture mechanics following the crack growth, series of short cracks, optimal design problem, etc. (e.g. Schiermein, Szabo [1986]).

5) Easy learning and robust performance.

The p and h-p version is very well responding to needs of this type. Nevertheless, further new features as dealing with plate shells, fabricated plates, e dimensional problems, nonlinear analysis need to be developed.

### 13. RELATION TO SOME OTHER METHODS

The p and h-p versions of the finite element method has a relation to todays form of the spectral method. The spectral method (see, e.g., Gottlieb, D. D., Hussaini, Orszag, [1984]) expands the solution of the problem in high-order Fourier or polynomial series, the coefficients of which are determined by weighted-residual projections. The spectral method is used in the fluid mechanics, for example, in the problem of transitional and low Reynold number, turbulent incompressible fluid flow in simple domains. Recently, the research of the spectral method is focused on extensions to more complex domains. The spectral method uses Galerkin type approach and is assuming that the conditions (2.5) holds. The spectral method is, with the use of polynomial approximation, closely related to the p-version with differences in applications and implementations. See Patera [1986] and the references given there. The pseudospectral method can be viewed as the Galerkin method with the numerical integration technique.

The spectral method was traditionally analyzed in the context with smooth solutions. Only recently the needs of coping with the singularities and geometries in the spirit of the h-p version is seen.



The p-version is also related to the Global Element Method by Delves Phillips [1980].

The p and h-p versions of the finite element method is relatively well developed and tailored for the needs of general problems in structural mechanics as we have been seen in previous sections. The family of spectral methods which is applied in the field of fluid mechanics has various aspects which are related to the p and h-p versions discussed in this paper.

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